Dynamic Curing and Network Design in SIS Epidemic Processes

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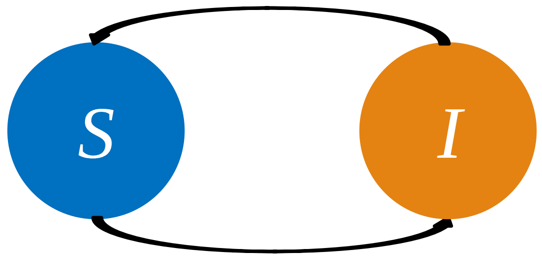
Susceptible-Infected-Susceptible (SIS) Model

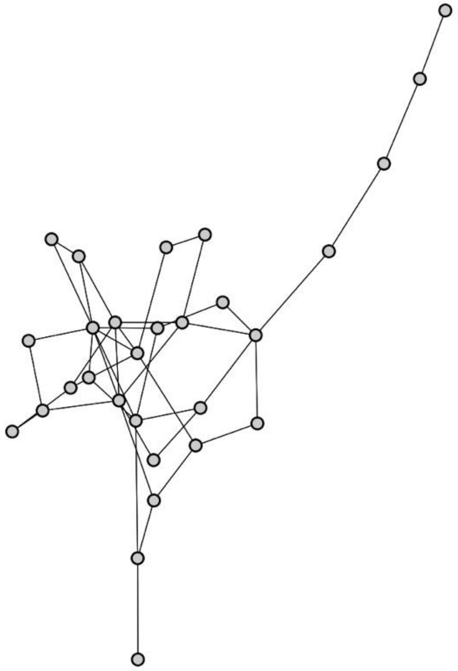
Modeling the spread of computer virus, information, human disease

[Kermack and McKendrick, 1927]
Compartmental Model

[Ganesh, Massoulié, Towsley, 2005]
Networked Markov Chain SIS Model

[Mieghem, Omic, Kooij, 2009]
Mean-field Approximations of SIS Model





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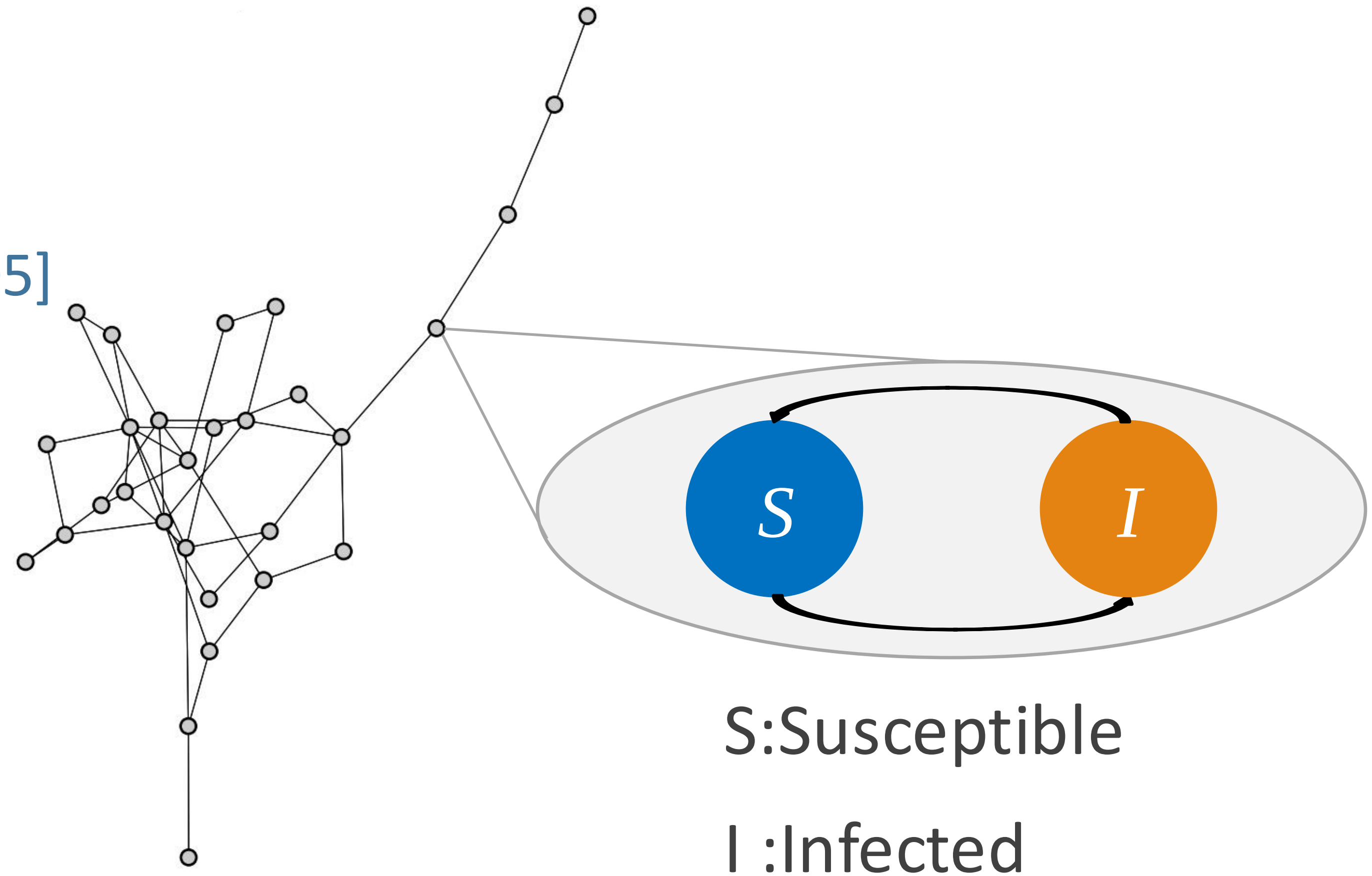
[Drakopoulos, Ozdaglar, Tsitsiklis, 2014] Dynamic Curing Policy



S:Susceptible

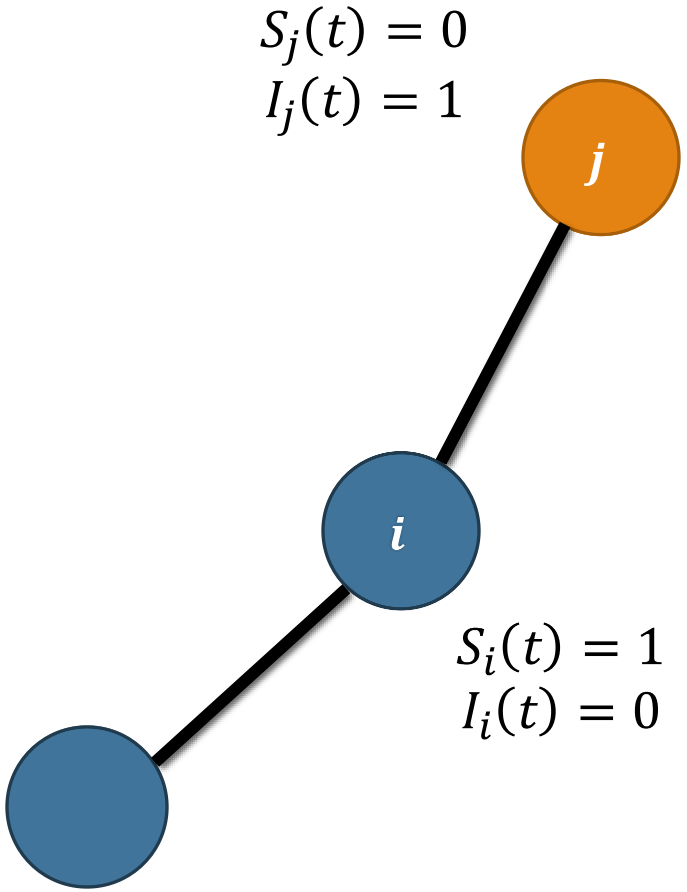
I :Infected

Continuous time Markov chain SIS Model

Contact network $G = (V, E)$

For any node $i \in V$ and time t

State: $S_i(t), I_i(t) \in \{0,1\}$



Continuous time Markov chain SIS Model

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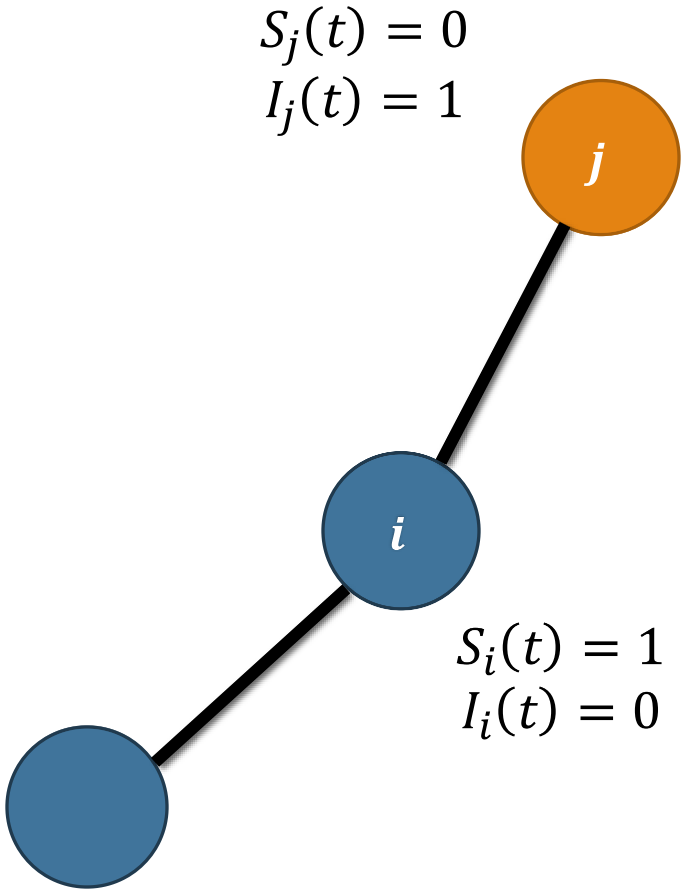
State: $S_i(t), I_i(t) \in \{0,1\}$

Infection rate of node v: $\sum_{(u,v)\in E} w_{uv} I_u(t)$

Curing rate: $\rho_v(t)$ for $v \in I(t)$

Curing budget: $\sum_{v\in V} \rho_v(t) \leq r$



Continuous time Markov chain SIS Model

Contact network $G = (V, E)$

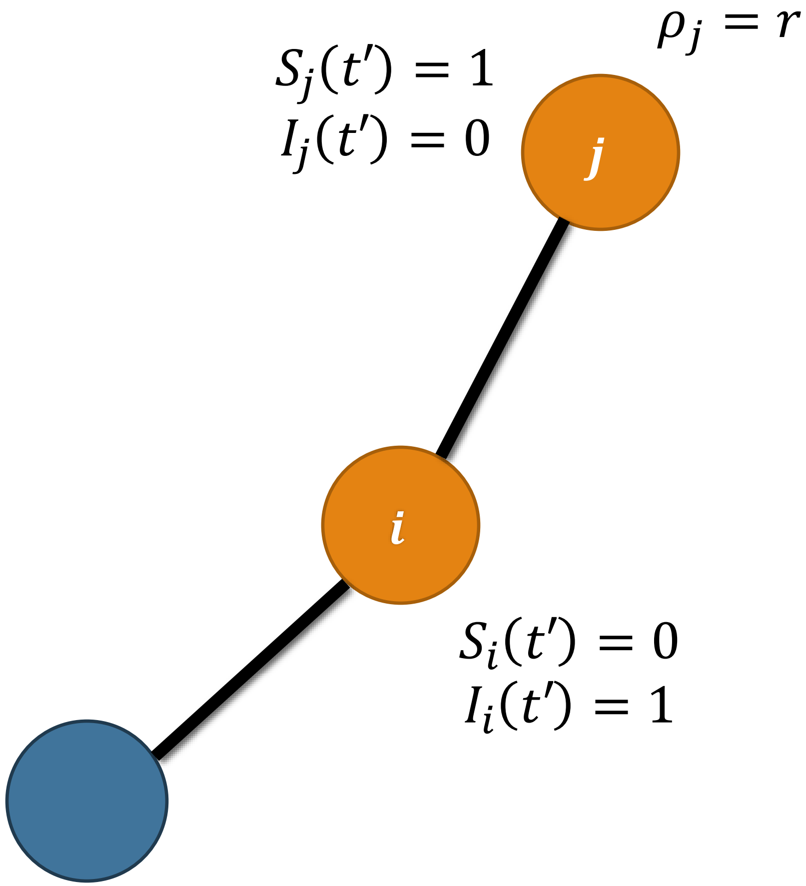
For any node $i \in V$ and time step t'

State: $S_i(t'), I_i(t') \in \{0,1\}$

Infection: $\sum_{(u,v)\in E^c} w_{uv},$

Recovery: $\rho_j = r$

Allocate curing budget to i or j?

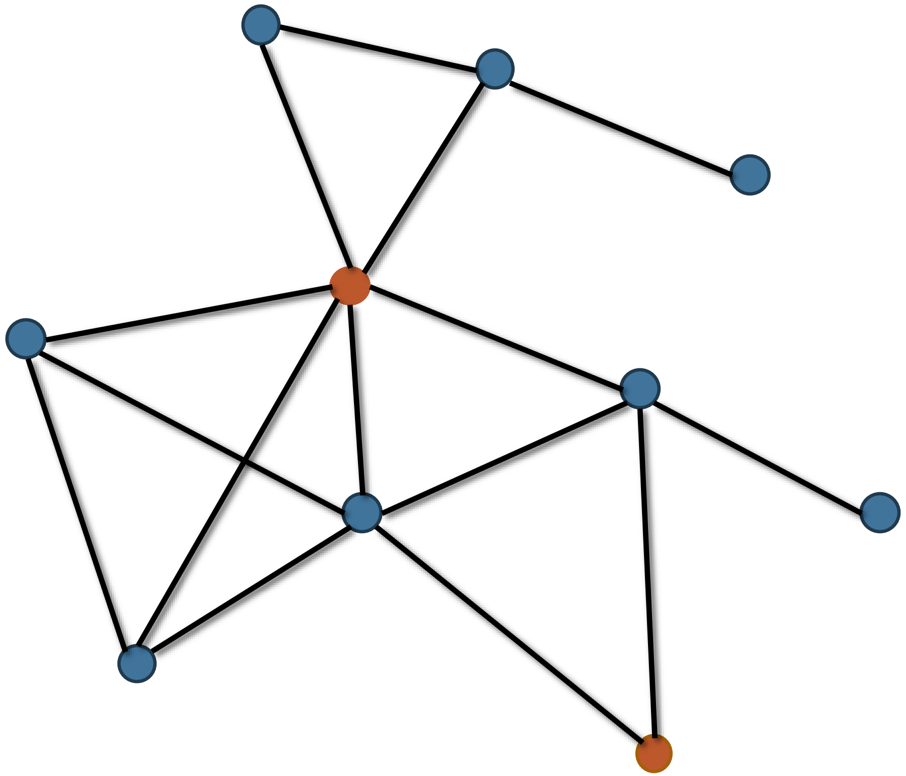


Dynamic Curing

Contact network $G = (V, E)$

Node states: $S_i(t)$ and $I_i(t)$ for all $i \in V$

Dynamic curing: allocate curing rates with minimum budget such that the expected extinction time is nearly linear.



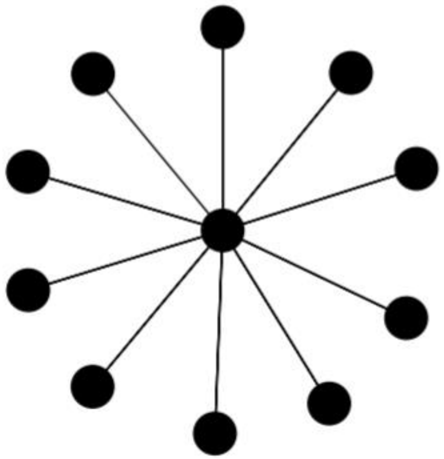
Related Concepts

- For $B \subset A$, a *crusade* $p(A, B)$ is a sequence of node sets $(p_0, \cdots, p_k)$, where $p_0 = A, p_k = B;\ p_i \subset p_{i-1}, |p_{i-1} \backslash p_i| = 1$ for $i \in [k]$
- $\mathcal{C}(A, B)$: set of all crusades from A to B
- Cut $c(A)$: number of edges connecting A and A^c
- Width $z(p)$: $\max_{0 \le i \le k} c(p_i)$
- Impedance $\delta(A)$: $\min_{\omega \in Cr(A, \emptyset)} z(\omega)$
- Cutwidth W: $\delta(V)$

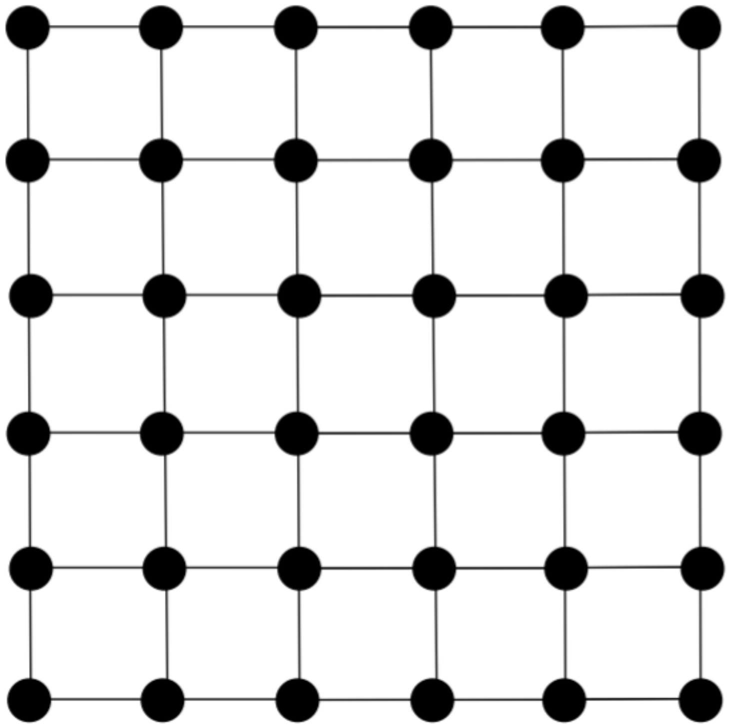
Cutwidth Examples

(a)

(c)



(b)



CURE Policy

[Drakopoulos, Ozdaglar, Tsitsiklis, 2014]

- Wait until $c(I_t) \leq r/8$. Let $A \leftarrow I_t$ right after waiting.
- Segment: Calculate the optimal crusade and the corresponding ordering of infected nodes $\{v_1, \cdots, v_{|A|}\}$, Allocate all curing resources to an arbitrary node in $D_t = I_t \backslash \{v_2, \cdots, v_{|A|}\}$ until
  1) D_t is empty, restart a segment
  2) $|D_t| \geq r/(8 \cdot d_{\max})$, start a new waiting period

Drawbacks:

Computational Complexity: Calculating the optimal crusade is NP-complete

Waiting period: no measure is taken

Optimal Crusade

Approximation algorithm for the optimal crusade

Divide and Conquer [Leighton and Rao, 1999]

Lemma: Given G, $\exists$ an algorithm which calculates a crusade p from V to $\emptyset$, such that

$$W \leq z(p) \leq O(\log^2 n)W.$$

Applying the algorithm to the subgraph supported on A,

Theorem: Given G and $A \subseteq V$, $\exists$ an algorithm which calculates a crusade p from A to $\emptyset$, such that

$$\delta(A) \leq z(p) \leq O(\log^{3/2} k \cdot \log\log k)\delta(A).$$

[BornStein, Vempala, 2003], [Feige, Lee, 2006]

Network Design

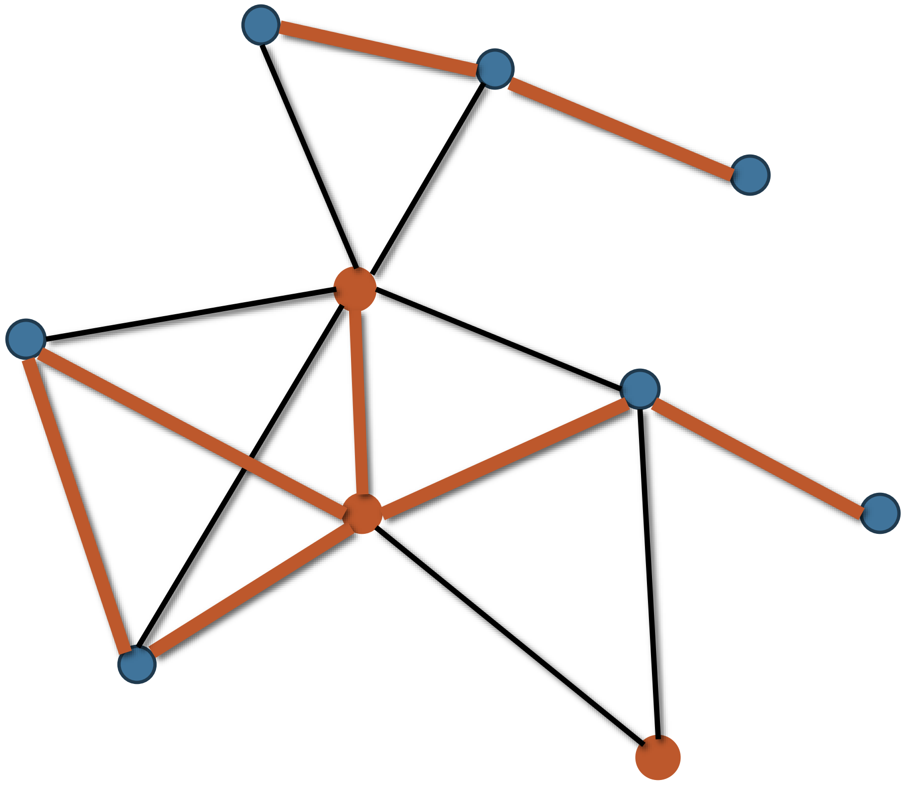
Nearly linear extinction time is guaranteed if

$$r \geq 4\alpha W \log^2 n\,.$$

Given a dynamic curing policy, find the *minimum edge weight reductions* such that $W \leq b$.

Instead of W, we minimize $z(p)$ after fixing the crusade p.

1. Fractional: Linear program
2. Integral: LP relaxation with a greedy rounding



Network Design

Given a Graph G with weight function w, a threshold b, find the edge reduction Δ_{uv} of each edge $uv \in E$ for the optimization program

$$\begin{aligned} \underset{\Delta}{\text{minimize}} \quad & \sum_{(u,v)\in E} \Delta_{uv}, \\ \textit{subject to} \quad & 0 \leq \Delta_{uv} \leq w_{uv}, \forall (u,v) \in E\,, \\ & z_{G'}(p) \leq b\,, \end{aligned}$$

G' is the modified graph with $\Delta'_{uv} = w_{uv} - \Delta_{uv}$.

Theorem: A polynomial time algorithm finds the optimal solution.

Network Design

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G' is the modified graph with $\Delta'_{uv} = w_{uv} - \Delta_{uv}$.

Theorem: A polynomial time algorithm finds a solution with an additive error of at most $kw_{\max}$.

Network Design

Consider the integral version with uniform edge weights

Equivalent to the *Interval Scheduling Problem on l machines* Problem

Earlist finish time first algorithm

Theorem: A greedy algorithm finds the optimal solution in $O(m \log m)$ running time.

Network Design with Arbitrary Curing Policy

- Minimizing the maximum cut of a (sub)graph.
- Define $\phi(\mathrm{A}) = \max_{Q \subseteq A} c(Q)$.
- Can be relaxed to a convex-concave minimax optimization problem
- Hyperplane rounding [Goemans and Williamson, 1995]

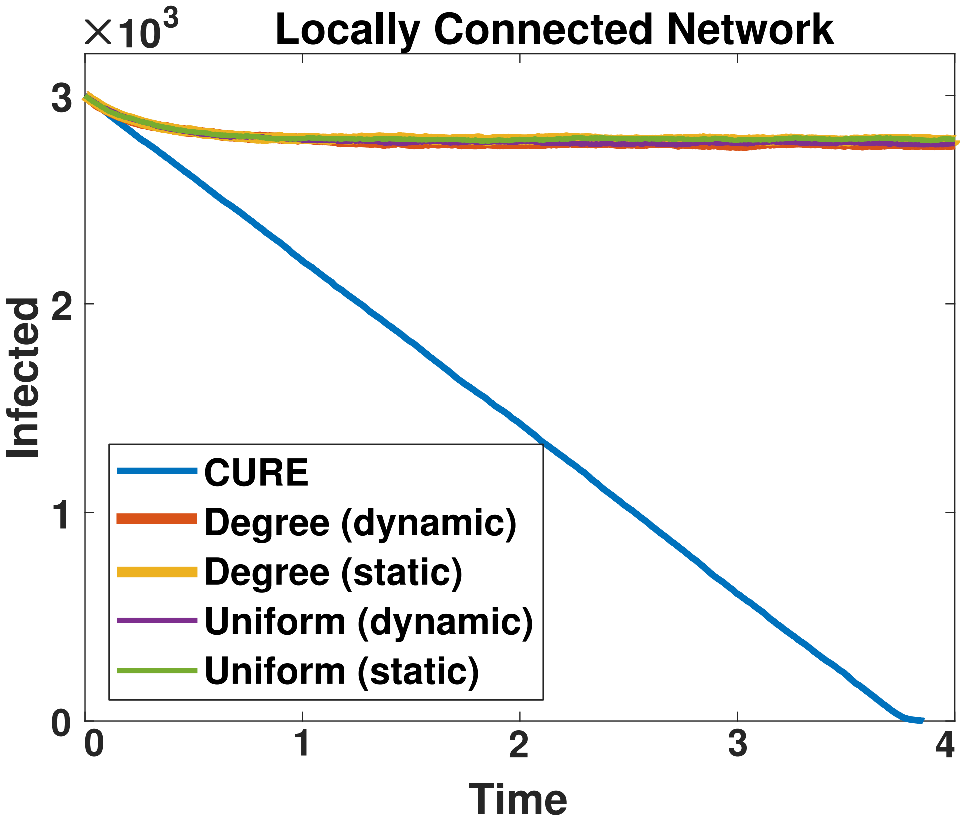
Theorem: Given a graph G and a set A, an edge weight reduction budget b, there exists an polynomial algorithm that finds an 1.44-approximation of the optimal $\phi_{G'}(A)$ with the same budget b.

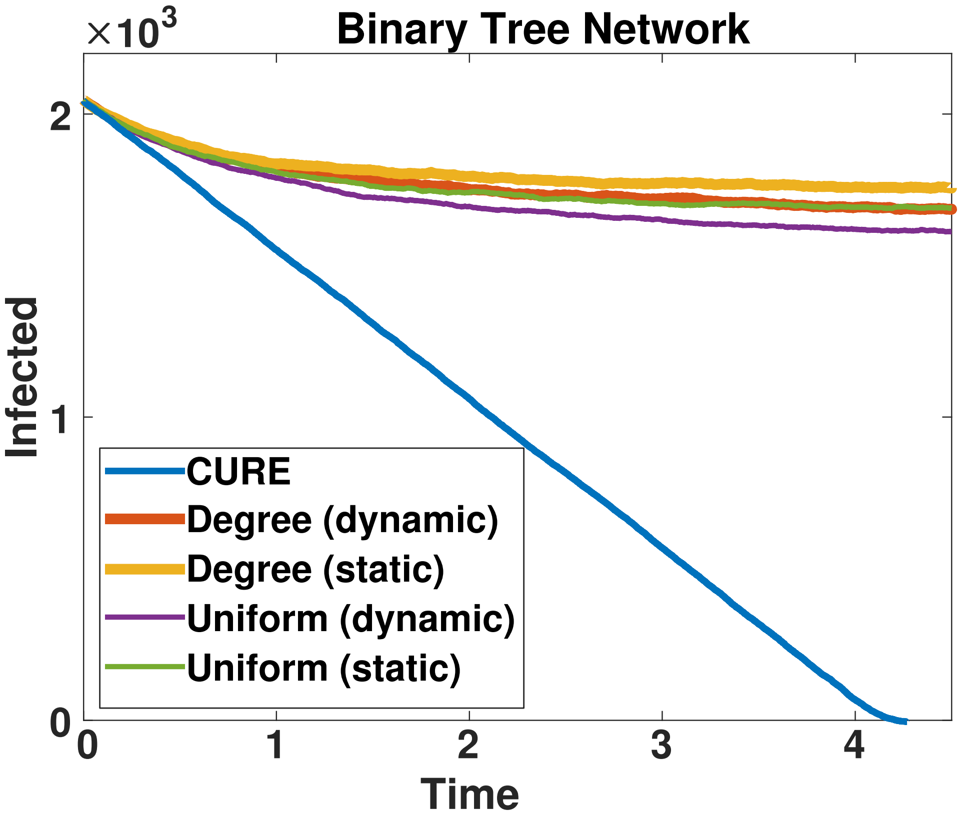
Numerical Simulations (Curing Policies)

CURE policy w/ the proposed approximation against Four Baselines

1. ***Uniform (static)***: *the curing budget is uniformly allocated to all nodes*
2. ***Degree (static)***: *proportional to the degree of nodes*
3. ***Uniform (dynamic)***: *uniformly allocated to all currently infected nodes*
4. ***Degree (dynamic)***: *proportional to the degree of currently infected nodes*

Numerical Simulations (Curing Policies)



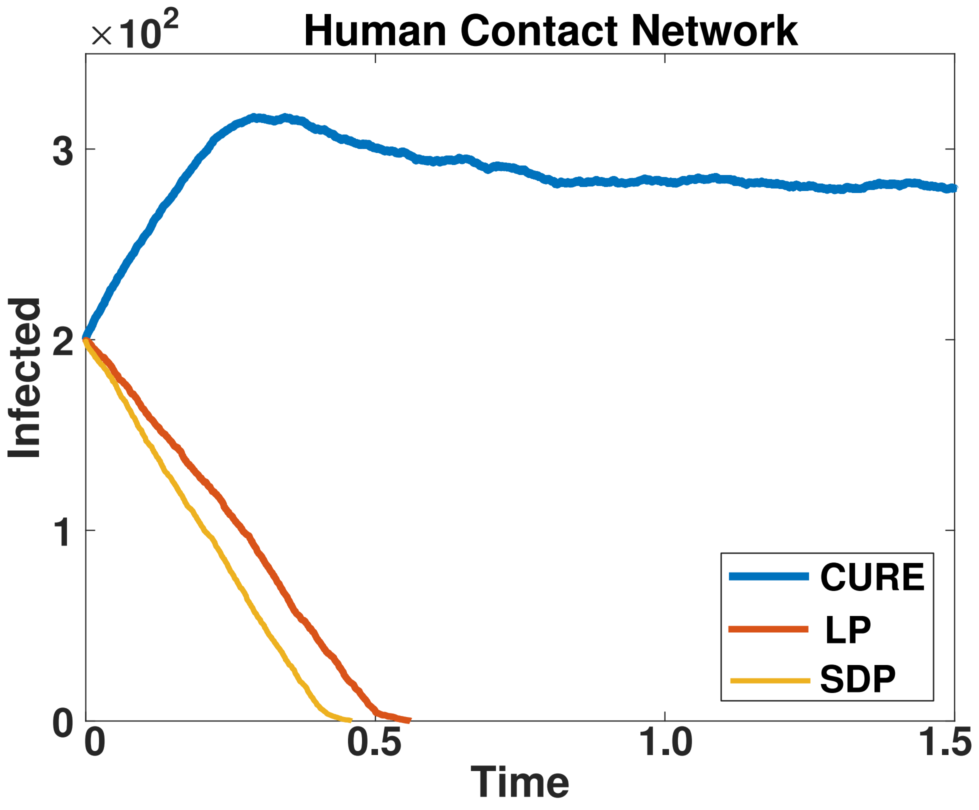


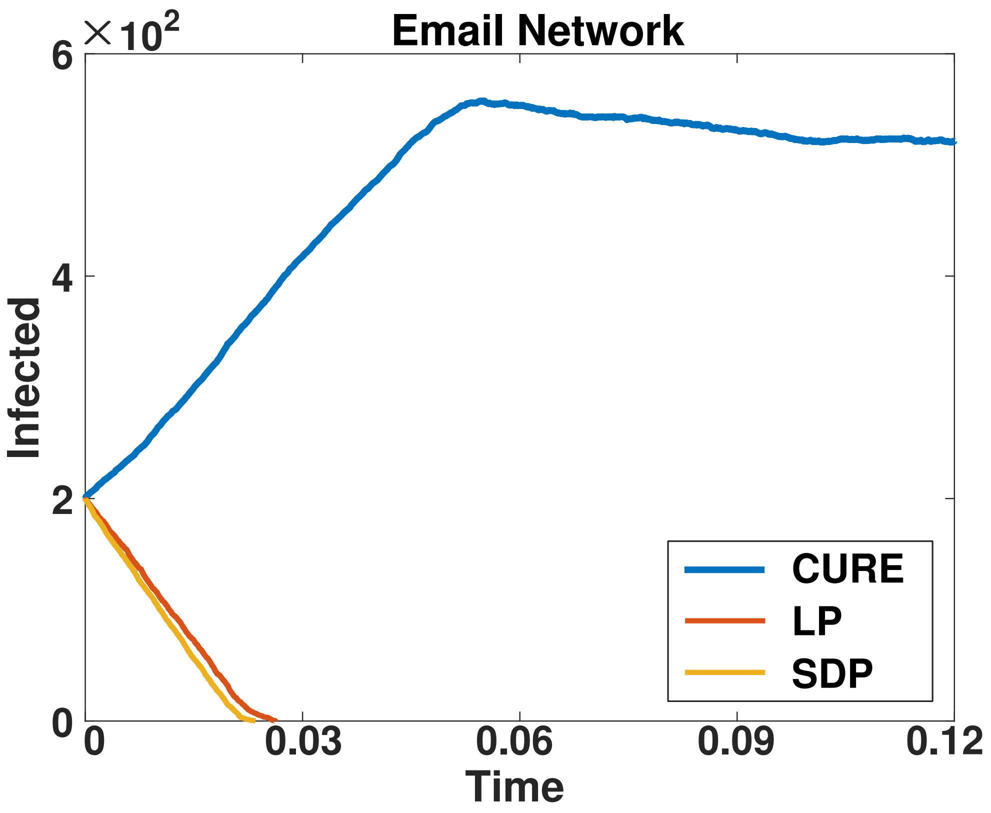
Numerical Simulations (Network Design)

We compare the following 3 cases:

1. CURE policy w/o network design
2. CURE policy augmented by LP-based edge weight reduction
3. Random Curing augmented by SDP-based edge weight reduction

Numerical Simulations (Network Design)





Future Directions

- Consider demographic fairness
- Minimizing impedance without fixing the crusade p
- Robustness of the algorithms
- Directed Networks

Thank you!

Questions