Disagreement and Polarization in Two-Party Social Networks

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Outline

1. Disagreement and polarization in consensus networks
   - Background on societal disagreement and polarization
   - French-DeGroot model and Friedkin-Johnsen model
   - Definitions of disagreement and polarization
2. Analysis and optimization
   - Preliminaries
   - Analyzing and optimizing the French-DeGroot model
   - Analyzing and optimizing the Friedkin-Johnsen model
3. Conclusion and future work

Disagreement and Polarization

In a social network, nodes are influenced by internal or external sources of polarizing opinions.

- *Disagreement*: the differences between neighbors.
- *Polarization*: the deviation of states from the system average.
- *Group consensus* in networks with communities.
- Trade-off between disagreement and polarization [1].

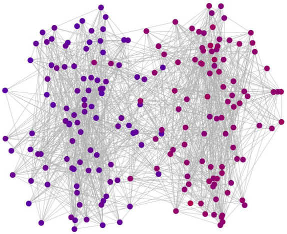


Figure: Opinions in a social network with external influence.

[1] Musco, C., Musco, C., and Tsourakakis, C.E. WWW'18.

French-DeGroot model

- Sources of polarizing opinions (leaders) are located in the network.
- Nodes are divided into leader set S and follower set F.

$$\begin{aligned} x_v(0) &= x_v^0, && v \in S\,, \\ \dot{x}_v(t) &= -\sum_{u \in N_v} w(u,v)(x_v(t) - x_u(t)), && v \in F\,. \end{aligned}$$

- $w(u,v)$: weight of the edge (u,v).
- We consider the case where $S = \{s_0, s_1\}$.
- The system has a unique steady state $\hat{x}$.

Friedkin-Johnsen model

- Sources of polarizing opinions are external.
- Every node is affected by external influence of both parties.

$$\begin{aligned}\dot{x}_v(t) = & \beta_v \kappa_v \cdot (1 - x_v(t)) + (1 - \beta_v)\kappa_v \cdot (0 - x_v(t)) \\ & + \sum_{u \in N_v} w(u,v)(x_u(t) - x_v(t)) \,,\end{aligned}$$

- $\kappa_v \geq 0$: susceptibility to persuasion[2] of node v.
- $\beta_v \in [0,1]$: preference of node v to opinion 1 over opinion 0.
- The system has a unique steady state $\hat{x}$.

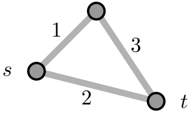
[2] Abebe, R., Kleinberg, J., Parkes, D., and Tsourakakis, C.E. KDD'18.

Definitions of Disagreement and Polarization

- Disagreement between nodes: $d(u,v) \stackrel{\text{def}}{=} w(u,v)(\hat{x}_u - \hat{x}_v)^2$.
- Disagreement in a network: $\mathcal{D} \stackrel{\text{def}}{=} \sum_{(u,v)\in E} d(u,v)$.
- Polarization in a French-DeGroot network:
  $\mathcal{P} \stackrel{\text{def}}{=} \sum_{u\in V} \left(\hat{x}_u - \frac{\sum_{u\in V} \hat{x}_u}{n}\mathbf{1}\right)^2$.
- Polarization in a Friedkin-Johnsen model:
  $\tilde{\mathcal{P}} \stackrel{\text{def}}{=} \sum_{u\in V} (\hat{x}_u - \alpha\mathbf{1})^2$, where $\alpha = \frac{\sum_{u\in V} \kappa_u \hat{x}_u}{\sum_{u\in V} \kappa_u}$ is the consensus value when $w(u,v) \to +\infty$ for all edges.
- Polarization-Disagreement Index: $\mathcal{I} = \rho\mathcal{D} + (1-\rho)\mathcal{P}$.
- Weighted Polarization-Disagreement Index: $\tilde{\mathcal{I}} = \rho\mathcal{D} + (1-\rho)\tilde{\mathcal{P}}$.

Outline

Graph Laplacians



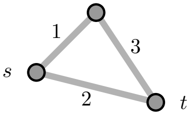
$$\Leftrightarrow \begin{pmatrix} 3 & -1 & -2 \\ -1 & 4 & -3 \\ -2 & -3 & 5 \end{pmatrix}$$

Coordinates ⇔ Vertices

Off-diagonal entries ⇔ Edges

- Laplacian $L_{u,v} \stackrel{\text{def}}{=} \begin{cases} \sum_{u \neq v} w(u,v) & u = v \\ -w(u,v) & u \neq v. \end{cases}$

Graph Laplacians



$$\Leftrightarrow \begin{pmatrix} 3 & -1 & -2 \\ -1 & 4 & -3 \\ -2 & -3 & 5 \end{pmatrix}$$

Coordinates $\Leftrightarrow$ Vertices

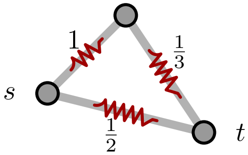
Off-diagonal entries $\Leftrightarrow$ Edges

- Laplacian $L_{u,v} \stackrel{\text{def}}{=} \begin{cases} \sum_{u \neq v} w(u,v) & u = v \\ -w(u,v) & u \neq v. \end{cases}$
- $L = \sum_{e \in E} w(u,v) b_{u,v} b_{u,v}^T$, where $b_{u,v} = e_u - e_v$.

$$\begin{pmatrix} 3 & -1 & -2 \\ -1 & 4 & -3 \\ -2 & -3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{pmatrix}.$$

- L is PSD as $x^T L x = \sum_{(u,v) \in E} w(u,v)(x_u - x_v)^2$.
- Defining pseudoinverse $L^\dagger$ by inverting all nonzero eigenvalues.

Resistance distance and biharmonic distance



$$r(s,t) = \frac{1}{\frac{3}{4}+2} = \frac{4}{11}\,\Omega$$

- The resistance distance is defined as the voltage difference between u and v when unit current is injected at u and extracted from v. $r(u,v) = b_{u,v}^{\top} L^{\dagger} b_{u,v}$, for any vertex pair u, v[3].
- The biharmonic distance[4] is defined on any vertex pair u, v:

$$d_B(s,t) = \sqrt{b_{u,v}^{\top} L^{2\dagger} b_{u,v}}\,.$$

[3]Klein, D.J. and Randić, M. (1993). Resistance distance. J. Math. Chem., 12(1), 81–95.

[4]Lipman, Y., Rustamov, R.M., and Funkhouser, T.A. (2010). Biharmonic distance. ACM Trans. Graph., 29(3), 27.

Analyzing the French-DeGroot model (I)

- Matrix form ($\hat{x}_S = \{0, 1\}$):

$$\dot{x}_S = \vec{0}\,, \qquad\qquad L = \begin{pmatrix} L_{S,S} & L_{S,F} \\ L_{F,S} & L_{F,F} \end{pmatrix}.$$
$$\dot{x}_F = -L_{F,F}x_F - L_{F,S}x_S\,,$$

- Steady state: $\hat{x} = -(L_{F,F})^{-1}L_{F,S}x_S = -(L_{F,F})^{-1}L_{F,s_1}$.

Lemma

In a two-party French-DeGroot model, let s_0 and s_1 be the leaders for opinion 0 and 1. The steady state $\hat{x}_v$[5] of node v is given by $\hat{x}_v = \dfrac{b_{v,s_0}^T L^\dagger b_{s_1,s_0}}{b_{s_1,s_0}^T L^\dagger b_{s_1,s_0}}$.

[5] Como, G. and Fagnani, F. (2016). From local averaging to emergent global behaviors: The fundamental role of network interconnections. Syst. Contr. Lett., 95, 70–76.

Analyzing the French-DeGroot model (II)

Theorem (Disagreement)

In the French-DeGroot model, the disagreement $\mathcal{D}$ in the considered opinion network is

$$\mathcal{D} = \frac{1}{b_{s_1,s_0}^T L^\dagger b_{s_1,s_0}} = \frac{1}{r_{s_1,s_0}} .$$

Theorem (Polarization)

The polarization $\mathcal{P}$ in the French-DeGroot opinion network is

$$\mathcal{P} = \frac{b_{s_1,s_0}^T L^{2\dagger} b_{s_1,s_0}}{(b_{s_1,s_0}^T L^\dagger b_{s_1,s_0})^2} = \left(\frac{d_B(s_1,s_0)}{r_{s_1,s_0}}\right)^2 .$$

By the Cauchy-Schwartz inequality, $\mathcal{P} \geq \frac{1}{2}$.

Network design: leader selection

Problem

In a French-DeGroot model, given the graph $G = (V, E, w)$ and a opinion leader s_0 for opinion 0 , choose a single opinion leader s_1 for opinion 1 such that $\mathcal{I} = \rho\mathcal{D} + (1 - \rho)\mathcal{P}$ (for a fixed ρ) is minimized.

- $\mathcal{D}$ is minimized when r_{s_0,s_1} is maximized.
- $\mathcal{P}$ is determined by $\frac{d_B(s_0,s_1)}{r_{s_0,s_1}}$.
- $O(n^3)$ algorithm (can be improved).

Network design: a robust structure (I)

Problem

In a French-DeGroot model, if the vertex set V is given, design the edge set E and weight function w of the graph $G = (V, E, w)$ with a cardinality constraint $|E| \leq k$ and a budget on total weight $\sum_{e \in E} w(e) \leq W$, such that $\max_{s_0, s_1} \mathcal{I}(s_0, s_1)$ is minimized.

- $\mathcal{P}(s_0, s_1)$ is minimized for all pairs of nodes s_0, s_1 iff G is a complete graph.
- $\mathcal{P}(s_0, s_1)$ will not change for any s_0, s_1 if edge weights are uniformly scaled.

Network design: a robust structure (II)

- Considering $k < \frac{n(n-1)}{2}$, $(|E| \leq k)$.

  Tool: spectral sparsification [6]

Theorem

In a French-DeGroot model, there exists a graph $H' = (V, \mathcal{E}, \bar{w})$ (and a polynomial time algorithm to find H') with $O(\frac{n}{\epsilon^2})$ edges that satisfies $\sum_{e \in E} \bar{w}(e) \leq W$, such that for any leaders s_0, s_1, $\mathcal{P} \in [\frac{1}{2}, (1+\epsilon)\frac{1}{2}]$.

- $\mathcal{D}(s_0, s_1)$ can be arbitrarily small for any s_0, s_1 if we multiply all edge weights with a sufficient small number a.

[6] Batson, J., Spielman, D.A., and Srivastava, N. (2012). Twice-ramanujan sparsifiers. SIAM J. Comput., 41(6), 1704–1721.

Network design: a robust structure (III)

Problem

In a French-DeGroot model, if the vertex set V is given, design the edge set E and weight function w of the graph $G = (V, E, w)$ with a cardinality constraint $|E| \leq k$ and a budget on total weight $W_\ell \leq \sum_{e \in E} w(e) \leq W_u$ such that $\max_{s_0, s_1} \mathcal{I}(s_0, s_1)$ is minimized.

- $\max_{s_0, s_1} \mathcal{D}(s_0, s_1)$ is also minimized when G is a complete graph (with all edges weighted $\frac{2W_\ell}{n(n-1)}$).
- $(1+\epsilon)$-approximation for optimal $\mathcal{D}$, $\mathcal{P}$, and $\mathcal{I}$ by sparsifying a complete graph.

Analyzing the Friedkin-Johnsen model

- Matrix form: $\dot{x}(t) = -(L+K)x(t) + BK\vec{1}$.
- Steady state:
$$\hat{x} = (L+K)^{-1}BK\vec{1}.$$

where B and K are diagonal matrices. $B_{v,v} = \beta_v$, $K_{v,v} = \kappa_v$.

Theorem

In the Friedkin-Johnsen model, the disagreement $\mathcal{D}$ is
$$\mathcal{D} = \widetilde{s}^T(L+K)^{-1}L(L+K)^{-1}\widetilde{s}\,,$$
the weighted polarization $\tilde{\mathcal{P}}$ is
$$\tilde{\mathcal{P}} = \widetilde{s}^T(L+K)^{-1}K(L+K)^{-1}\widetilde{s}\,.$$
where $\widetilde{s} = P^T BK\mathbf{1}$, $P = I - \frac{\vec{1}\vec{\kappa}^T}{\sum_{v\in V}\kappa_v}$.

Network design: designing weights

Problem

In the Friedkin-Johnsen Model, given the node set V and edge set E, design the edge weights $w(e) \in [\ell, p]$ with budget $\sum_{e \in E} w(e) \leq W$ such that the quantity $\tilde{\mathcal{I}} = \rho\mathcal{D} + (1-\rho)\tilde{\mathcal{P}}$ (for a fixed ρ) is minimized.

Theorem

The weighted Polarization-Disagreement Index $\tilde{\mathcal{I}} = \frac{1}{2}\mathcal{D} + \frac{1}{2}\tilde{\mathcal{P}}$ is a convex function of the edge weights $\vec{w}$ of the graph, where the entries of the vector $\vec{w}$ are defined as $\vec{w}_e = w(e)$, $e \in E$.

- Proof: the epigraph of $f(Y, \widetilde{s}) = \widetilde{s}^T Y^{-1} \widetilde{s}$, where $Y = (L + K)$, is a convex set in both Y and s.
- Polynomial time solvable by using a standard SDP solver.

Network design: designing preferences

Problem

Consider the Friedkin-Jonson Model for $G = (V, E, w)$, and an integer k. Assume the preference of node $v \in V$ is either 0 or 1, then V can be partitioned into two disjoint sets P_0 and P_1, where $\beta_v = 0$ for $v \in P_0$ and $\beta_v = 1$ for $v \in P_1$. Flip the preferences β_v of all nodes in Q, where $Q \subseteq P_0$ (or exclusively $Q \subseteq P_1$), $|Q| \leq k$, such that $\tilde{\mathcal{I}}$ is minimized.

Theorem

The indices $\mathcal{D}$, $\tilde{\mathcal{P}}$, and $\tilde{\mathcal{I}}$ are all convex functions of the vector $\vec{\beta} \in \mathbb{R}^n$, where the entries of the vector are defined as $\vec{\beta}_v = \beta_v$ for all $v \in V$.

Network design: designing preferences

- Heuristic algorithm: convex relaxation with ℓ_1 regularization.

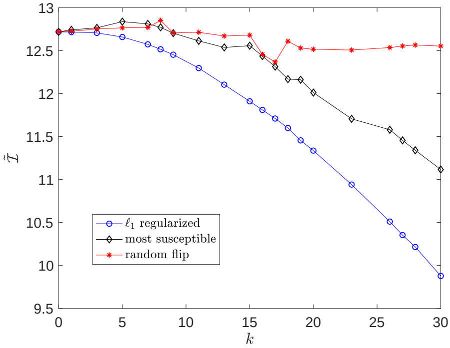


Figure: The value of $\tilde{\mathcal{I}}$ ($\rho = 1/2$) and number of flips k we get from the ℓ_1 regularized optimization, compared with choosing the k most susceptible nodes in P_0, and choosing random nodes in P_0.

Conclusion and future work

- Conclusion
  1. Disagreement and polarization in French-DeGroot and Friedkin-Johnsen networks.
  2. Analysis and optimization.
  3. Analytical and numerical examples.
- Future work
  1. Directed networks.
  2. Top-k leader selection.
  3. Optimizing susceptibility of persuasion.

Thank you!